Exercises for 'Topics in complex analysis'

(08/10/2025)

H 5.1 (Absolute convergence of infinite products)
Let $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ be a sequence. We say that the infinite product $\prod_{j=1}^{\infty} a_j$ converges absolutely if there exists $j_0 \in \mathbb{N}$ such that $a_j \notin (-\infty, 0]$ for all $j \geq j_0$ and $\sum_{j\geq j_0} |\log(a_j)| < +\infty$.

a) Show that the absolute convergence of an infinite product implies its convergence.

b) Show that an infinite product of the form $\prod_{j=1}^{\infty}(1+b_j)$ converges absolutely if and only if $\sum_{j=1}^{\infty} |b_j| < +\infty$.

H 5.2 (Examples of infinite products)
Examine if the following infinite products exist in the sense of Definition 3.1. If so, calculate their value.

a) $\prod_{n=1}^{\infty}\left(1-\frac{1}{(n+1)^2}\right)$ b) $\prod_{n=1}^{\infty}\left(1-\frac{1}{n}\right)$ c) $\prod_{n=3}^{\infty}\frac{n^2-4}{n^2-1}$ d) $\prod_{n=1}^{\infty}\frac{(1+n^{-1})^2}{1+2n^{-1}}$

Hint: In all examples you can directly calculate the value of the partial products.

H 5.3 (A class of diverging products)
Let $\{a_j\}_{j\in\mathbb{N}} \subset [0, +\infty)$ be a sequence such that $\sum_{j=1}^{\infty} 1 - a_j = +\infty$. Show that $\lim_{n\to\infty} \prod_{j=1}^{n} a_j = 0$.

Hint: Use that $t \leq \exp(t-1)$ for all $t \in \mathbb{R}$.

H 5.4 (A useful criterion for the convergence of infinite products)
Let $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$. Assume that $\sum_{j=1}^{\infty} |a_j|^2 < +\infty$. Show that $\prod_{j=1}^{\infty}(1+a_j)$ converges if and only if $\sum_{j=1}^{\infty} a_j$ converges. Conclude that the infinite product $\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)$ converges only for $z = 0$.